Co-funded by the Erasmus+ Programme of the European Union

Water Management Optimization Problems

Basic Concepts of Linear Programming.

Application of Classic Optimization Tasks in Water Resources Management

Assoc. Prof. Petar Filkov
University of Architecture, Civil Engineering and Geodesy - Sofia

Sofia winter School 29 November – 10 December 2021

Date: 2 December – Lecture 7.

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

University of Nis

www.swarm.ni.ac.rs

Strengthening of master curricula in water resources management for the Western Balkans HEIs and stakeholders

Project number: 597888-EPP-1-2018-1-RS-EPPKA2-CBHE-JP

1. Optimization

- **Optimization** is an activity, which has an aim to determine *the best possible solution*, at a given criterion and under given circumstances (conditions)
- The best possible solution is the *optimal solution* or *optimum*
- The criterion is called *Objective Function* (OF) – a mathematical expression which shows if the optimum is achieved
  - According to the meaning of the OF the *optimum* is an extremum of the function – it is either **minimum**, or **maximum** of OF.
- The circumstances or conditions under which the optimum is found are called *constraints*.
- Thus, optimization means finding the minimum or maximum of the OF under given set of constraints.

1. Optimization

- Mathematical point of view
  - Optimization can be done in each case, when the number of unknown variables ***n*** is greater than the number of equations (or inequalities) ***m*** which set interrelation between n variables.
- For example, let's have:
  - n – number of variables;
  - m – number of equations, which link these n variables.
  - When $n < m$, then there is no solution, or only one solution
  - When $n = m$, then there is only one possible solution
    - ✓ We cannot say if the solution is good or bad – it is only one
  - When $n > m$, then there are infinite number of solutions
    - ✓ In that case we may look for the best solution, according to some criteria

1. Optimization

- When $n > m$, then:
  - The equations (or inequalities) which set the interrelation between variables are *the constraints*.
  - In order to find the best possible solution, i.e. optimum solution, we have to specify the OF
  - The OF has to be the function of the variables.
  - There are three kinds of variables:
    - ✓ **decision variables** (non-basic) variables – their number is $\boldsymbol{k = n - m}$
    - ✓ **basic variables** – their number is m
      - These are the variables which can be expressed as function of decision variables by means of the available m equations or inequalities.
    - ✓ **slack variables** – additionally introduced variables, which have the aim to convert inequalities to equations
      - e.g. inequality $a.X_1 + b.X_2 \leq D$ turns to: $a.X_1 + b.X_2 - t_1 = D$, as $t_1 \geq 0$

1. Optimization

- The optimization problem is defined as:

  Find the min/max of the OF

  $$Z = f(X_i),$$

  subject to:

  $$\sum_i a_{ij} X_i \overset{\le}{\underset{\ge}{}} b_j$$

  where a_{ij} are the multiplication coefficient of i-th variable in j-th constraint.

  b_j – is the j-th constraint value

- When *all equations* (and inequalities) *are linear* in respect to variables X_i, and also the *OF is a linear function* of variables X_i, then we speak about linear optimization or **linear programming**

1. Optimization

- The following requirements have to be fulfilled, in order to have optimization problem:
  - **Subject to optimization**
    - ✓ In Water Resources Management (WRM) this is Water Management System (WMS) – water supply, irrigation or hydro-power system
  - **Manageability of the subject**
    - ✓ In WRM the WMS, as well as the river runoff are manageable
  - **Optimization criteria**
    - ✓ This is the Objective Function
  - **Optimization method**
    - ✓ There are different methods – analytical, numerical, graphical or experimental
      - – the analytical and experimental methods are not suitable for WRM

1. Optimization

- The following types of optimizations are present:
  - **Static optimization** – when the system subject to optimization is considered (and analyzed) in a static (steady) state.
  - Dynamic optimization (**Dynamic Programming**)
    - ✓ when (some of) variables in the OF are time-dependant, i.e. they change over time, or
    - ✓ when it is necessary to analyze the systems in several steady states over time.
  - Linear optimization (**Linear Programming - *LP***) – when all constraints and the OF are in linear relation with variables
  - Non-linear optimization (**Non-linear Programming**) – when some or all of the constraints and/or OF are in non-linear relation with variables.

1. Optimization

- **Water Resources Management** is actually an optimization problem

The water resources management aims determination of the optimal use of water resources, in accordance with assumed optimization criterion, under set of constraints

- *Subject* to optimizations are water management systems
- *Constraints* are different limitations regarding the size of the systems, the availability of water resources, etc.
- Optimization criterion can be economic, technical and economic, etc.
- *Use of water resources has to be determined.*

2. Linear Programming (LP)

- The optimization problem is defined as:

For the OF: $Z = C_0 + C_1X_1 + C_2X_2 + \ldots + C_nX_n$ find min (or max),

Subject to (constraints):

$$\begin{vmatrix} a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n \geq b_1 \\ a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n \leq b_2 \\ \ldots \\ a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{m,n}X_n \geq b_m \end{vmatrix}$$

There are additional constraints in lots of problems – all variables should be non-negative

$$X_i \geq 0$$

- In Linear Programming the min/max of the OF is conditional!
  - it depends of the constraints!

2. Linear Programming (LP)

- Graphical Interpretation of LP Problems
- When the number of decision variables is $k = 2$, then the solution can be presented in a 2-dimensional space, i.e. in a plane.
  - Let $k = n - m = 2$, and the decision variables are X_1 and X_2.
  - Then all m basic variables can be expressed as function of decision variables X_1 and X_2.

$$
\left|\begin{array}{l}
X_3 = \alpha_{31} X_1 + \alpha_{32} X_2 + \beta_3 \\
X_4 = \alpha_{41} X_1 + \alpha_{42} X_2 + \beta_4 \\
\ldots \\
X_m = \alpha_{m1} X_1 + \alpha_{m2} X_2 + \beta_m
\end{array}\right.
$$

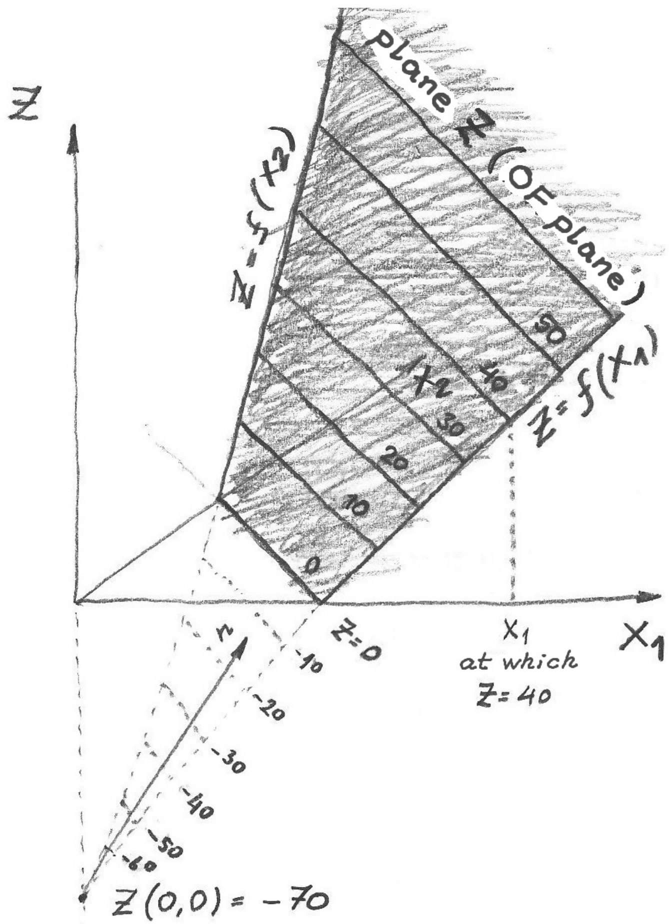
  - The OF is also expressed as function of X_1 and X_2

$$Z = \gamma_0 + \gamma_1 X_1 + \gamma_2 X_2$$

2. Linear Programming (LP)

- Graphical Interpretation of LP Problems
- For thar case, the equation of the OF is the equation of the plane in the coordinate system $X_1 X_2 Z$

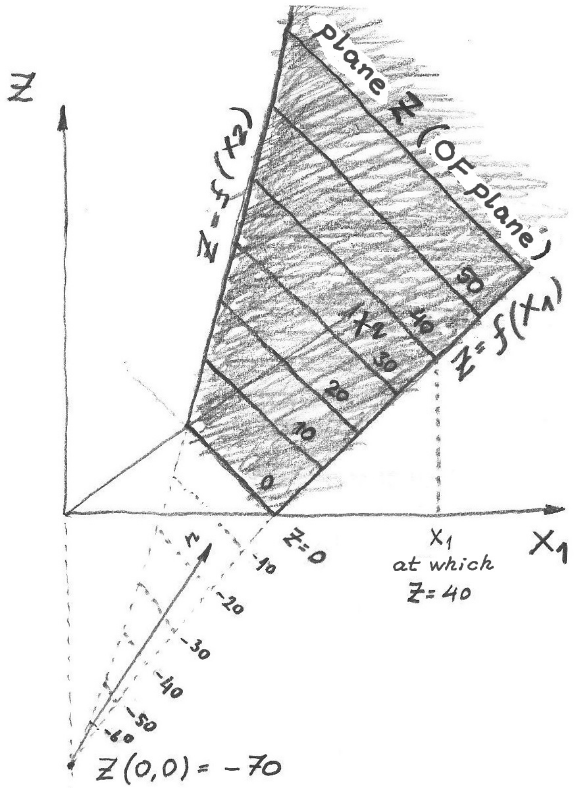
$$Z = \gamma_0 + \gamma_1 X_1 + \gamma_2 X_2$$

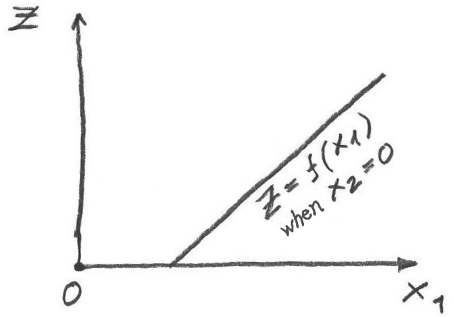


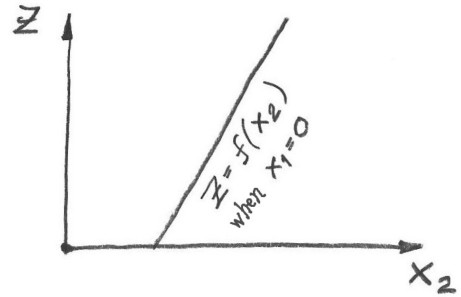
- ➢ The position in space $X_1 X_2 Z$ depends on:
  - ✓ the presence or absence and on the value of the coefficient γ_0,
  - ✓ the coefficients γ_1 and γ_2 which show the slopes towards the axes X_1 and X_2 – the bigger are the coefficients γ_1 and γ_2 , the steeper are the slopes.

2. Linear Programming (LP)

- Graphical Interpretation of LP Problems
  - The intersection between OF and the plane $X_1 0Z$ is a straight line
    - Its equation is obtained when value of $X_2 = 0$ is substituted in OF
  - The intersection between OF and the plane $X_2 0Z$ is a straight line
    - Its equation is obtained by when value of $X_1 = 0$ is substituted in OF

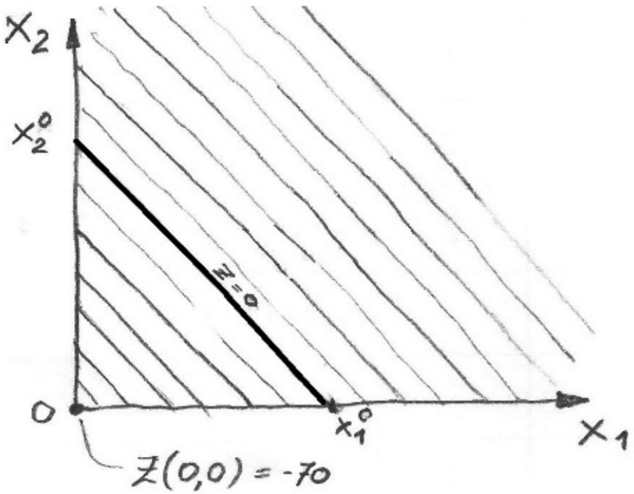


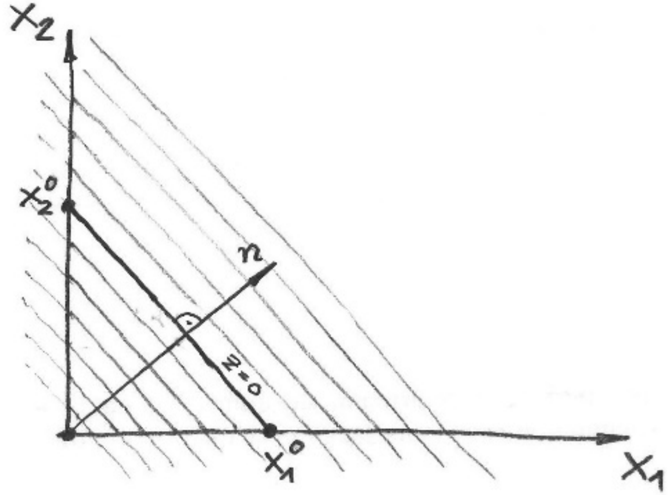




2. Linear Programming (LP)

- Graphical Interpretation of LP Problems
  - ➢ If we consider only 2D space, the OF can be presented in the plane $X_1 0 X_2$ similarly to terrain shape on a topographic (contour) map.





  - ➢ The constant values of OF can be presented as straight contours
    - ✓ On the left picture above the thick line is intersection between OF plane and $X_1 0 X_2$ plane
  - ➢ The biggest slope of OF plane towards $X_1 0 X_2$ plane is in direction of a line n (see right picture above)

2. Linear Programming (LP)

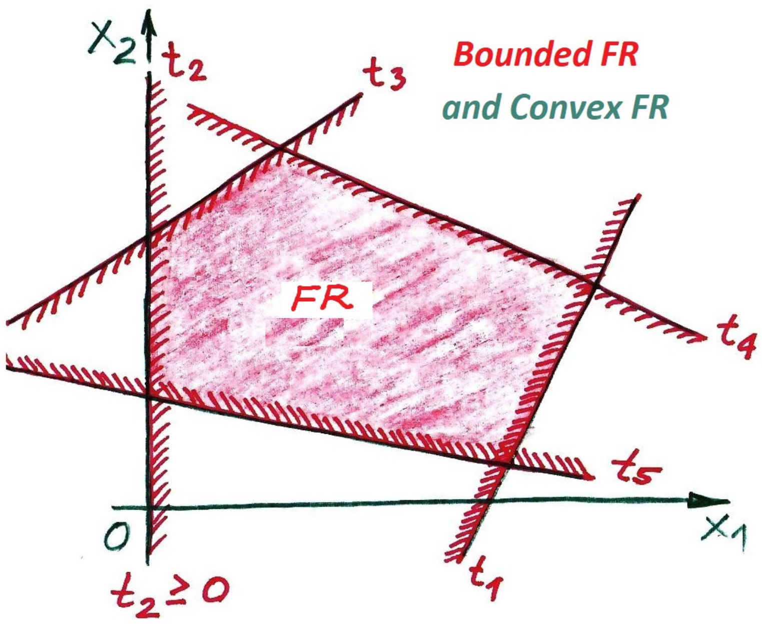
- Graphical Interpretation of LP Problems
  - Each constraint actually represent the equation of a plane, which crosses the $X_1 0 X_2$ plane

$$\begin{vmatrix} X_3 = \alpha_{31} X_1 + \alpha_{32} X_2 + \beta_3 \\ X_4 = \alpha_{41} X_1 + \alpha_{42} X_2 + \beta_4 \\ \ldots \\ X_m = \alpha_{m1} X_1 + \alpha_{m2} X_2 + \beta_m \end{vmatrix}$$

  - If the constraint is of type *equation*, by substituting any basic variable $X_i = 0$, we will get as result the equation of a line in $X_1 0 X_2$ plane.
    - ✓ All the points along the line are points, which mark *feasible solution*

2. Linear Programming (LP)

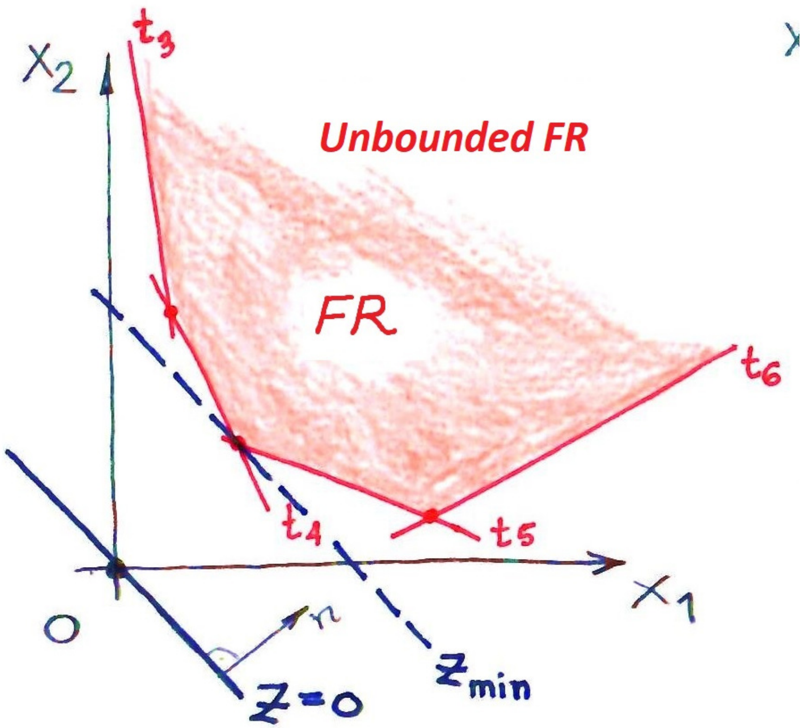
- Graphical Interpretation of LP Problems
  - If the constraint is of type *inequality*, by substituting any basic variable $X_i = 0$, and if we assume that this is equation, we will get as a result the equation of the line in $X_1 0 X_2$ plane.
    - ✓ Because of the inequality, this line splits $X_1 0 X_2$ plane in tow half-planes – one is an area with feasible solutions, and the other one – with non-feasible solutions



- ✓ All constraints create an area in the plane $X_1 0 X_2$
- ✓ If all the constraints are *equations*, then along all lines we have points, which show feasible solutions.
- ✓ If all the constraints are *inequalities*, then the region, which satisfies all inequalities is so-called ***feasible region***, or ***feasible set***.

2. Linear Programming (LP)

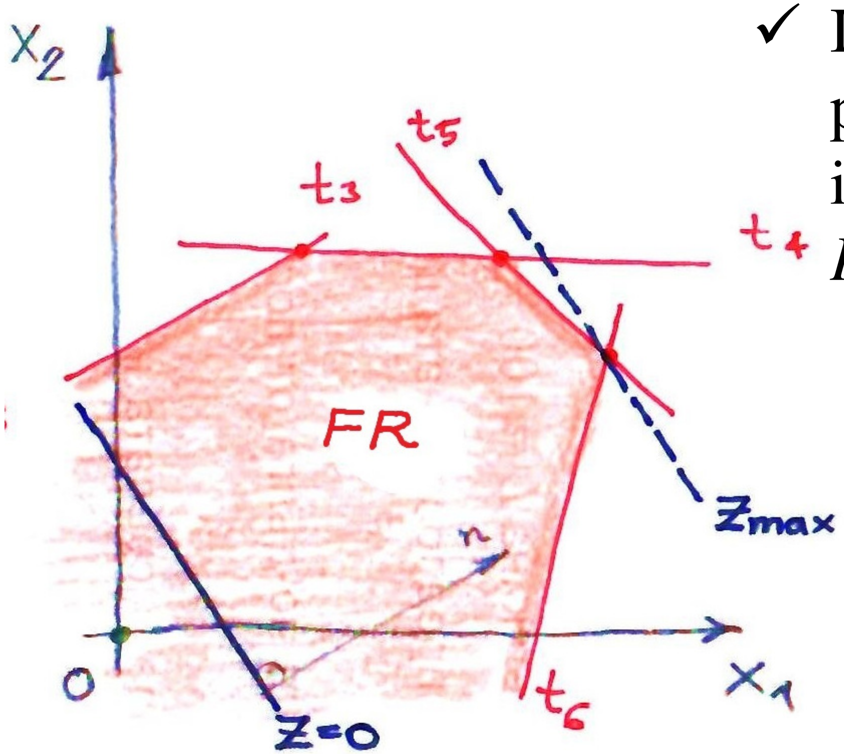
- Graphical Interpretation of LP Problems
  - Sometimes, because of number and type of inequalities, the feasible region may not be a compact figure, but it may consist a part of the plane $X_1 0 X_2$.



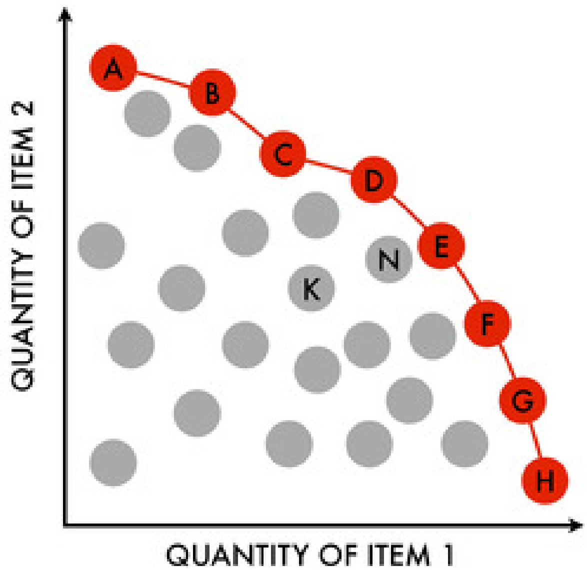
- It that case it is said that the region is ***unbounded***
- If *feasible region* is *unbounded*, it is important to have the borders of this region facing the origin of the coordinate system $X_1 0 X_2$, if we look for minimum of the OF.
  - ✓ This also means that the origin of the coordinate system is outside of the feasible region

2. Linear Programming (LP)

- Graphical Interpretation of LP Problems
  - If we look for maximum of the OF, it is important to have the feasible region restricted from Northeast, i.e. the origin of the coordinate system $X_1 0 X_2$ to be in the feasible solution region.

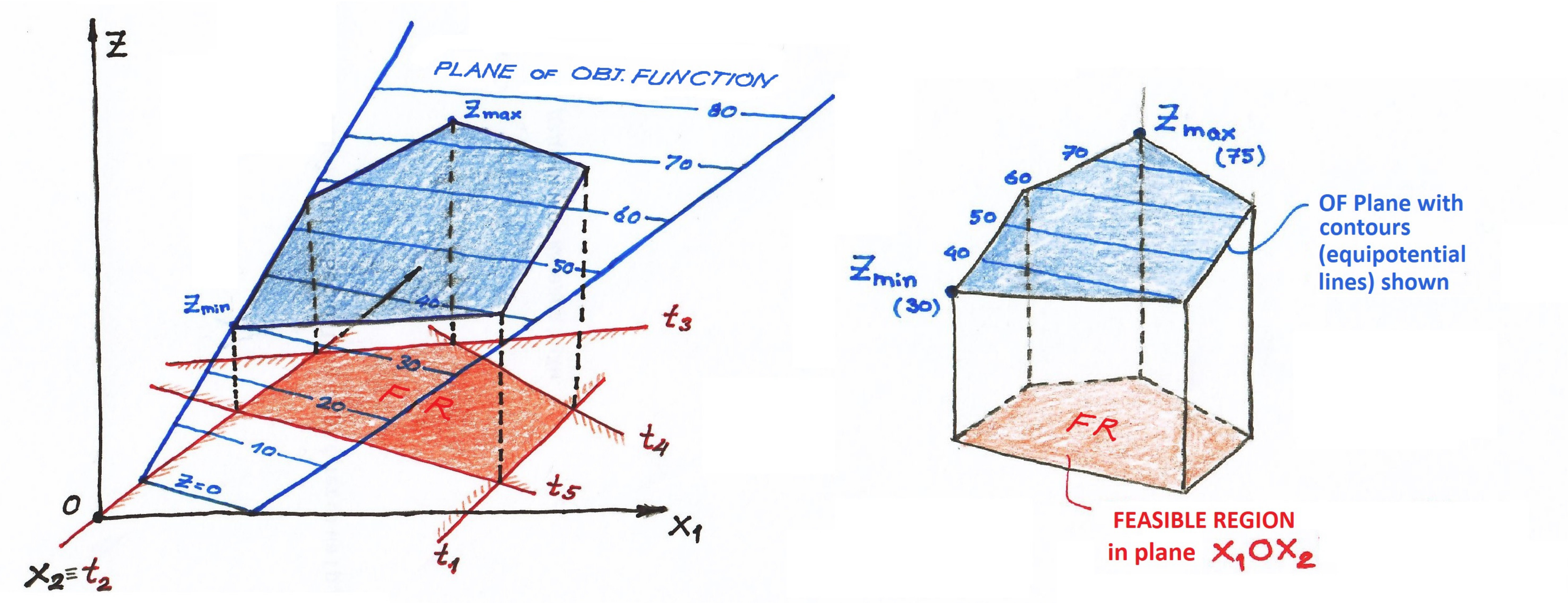


  - ✓ In case of unbounded feasible region, the polyline which marks the border of the region is known as *Pareto front*, *Pareto frontier* or *Pareto set*.



2. Linear Programming (LP)

- Graphical Interpretation of LP Problems
  - If the problem is presented in 3D space, then the following picture can be drawn

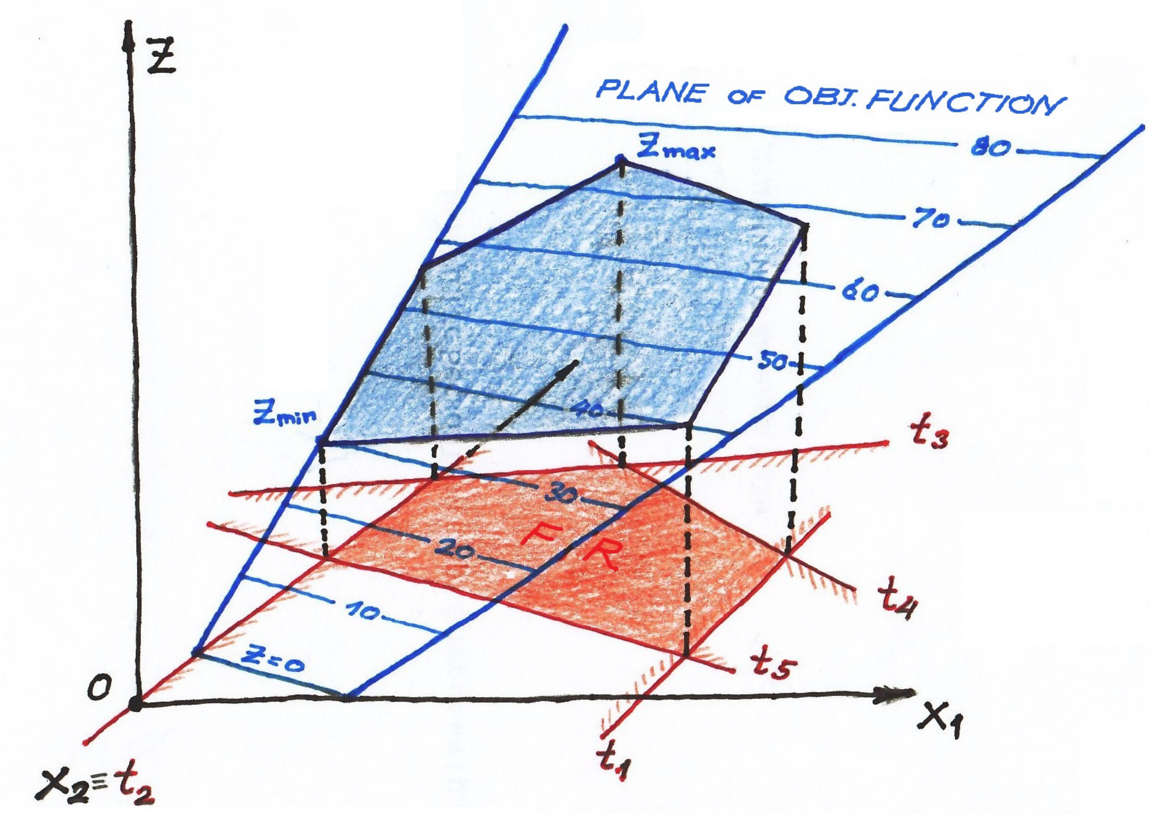


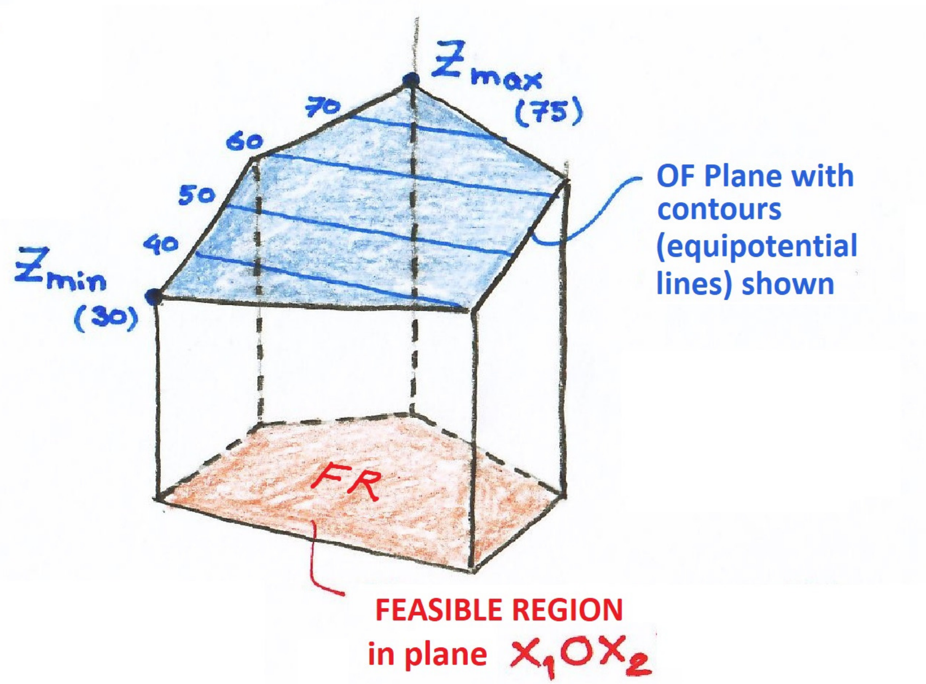
    - ✓ If the corners of the feasible region are projected towards the plane of the OF, we will get a polygon shape.

Co-funded by the Erasmus+ Programme of the European Union

2. Linear Programming (LP)

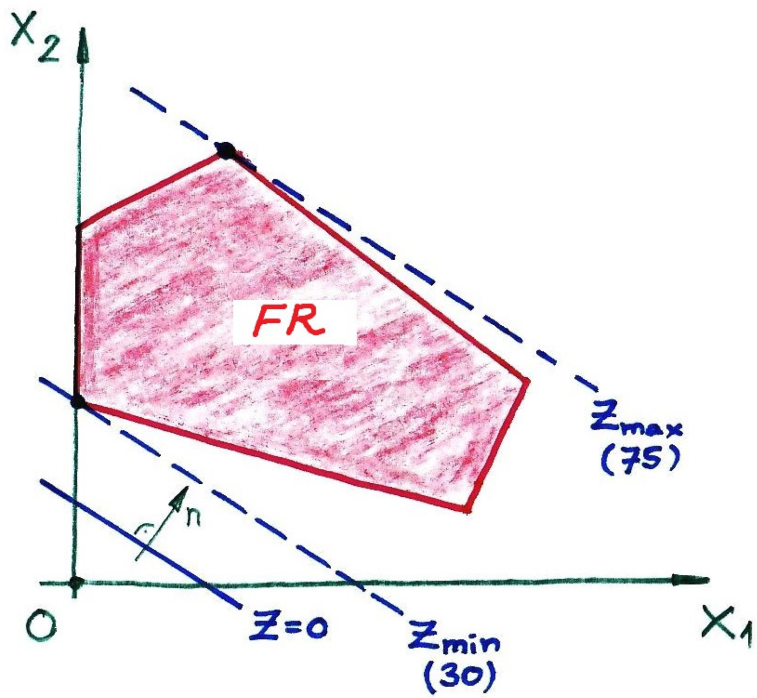
- Graphical Interpretation of LP Problems
  - It is seen that the maximum or minimum of the OF is found at one of the polygon corners in the plane of the OF





2. Linear Programming (LP)

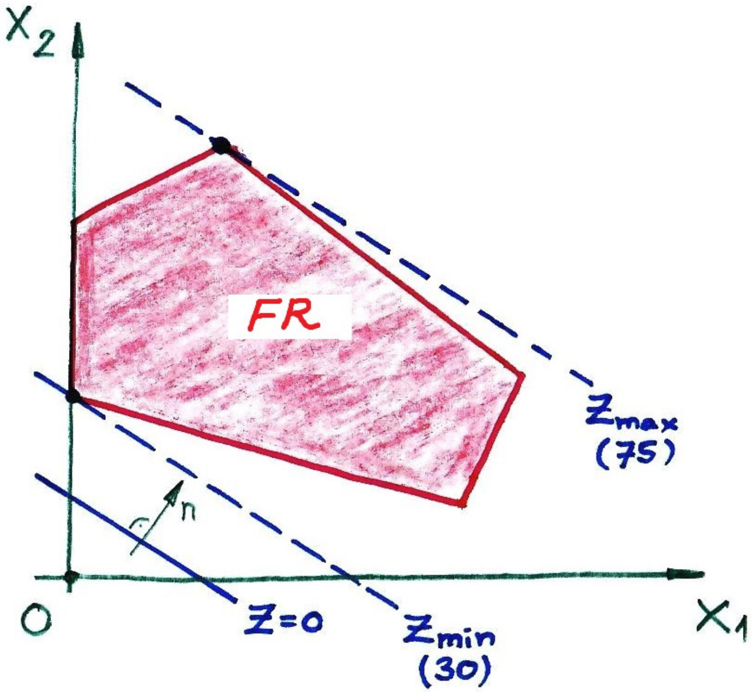
- Graphical Interpretation of LP Problems
  - In case of a 2D representation of the problem, all lines are drawn in the plane $X_1 0 X_2$.
  - The OF is presented also with lines – projection of equipotential lines of the OF over the plane $X_1 0 X_2$.



- ✓ The extremum (min or max) can be found if we move the line of the OF at value $Z = 0$ (this is intersection between plane $X_1 0 X_2$ and the plane of the OF) in normal direction towards the feasible region.
- ✓ The minimum or maximum ***always*** will be at some of the corners of the feasible region.

2. Linear Programming (LP)

- Graphical Interpretation of LP Problems
  - The shape of the feasible region can be changed, if one or more of the constrains are changed
  - Then the maximum or minimum is changed, due to change of the position of corners of the feasible region in the plane $X_1 0 X_2$.

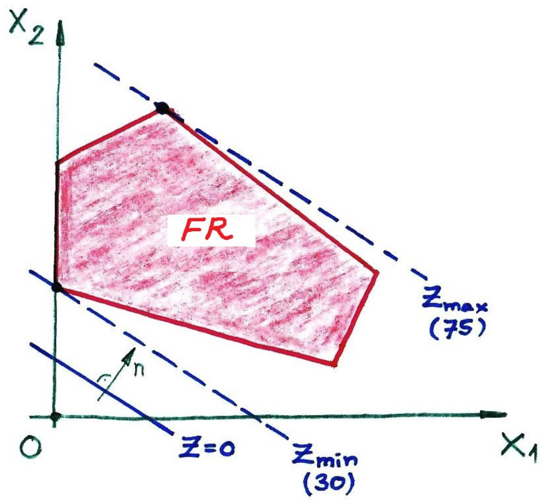


  - This is the reason why it is said that the minimum or the maximum of the OF in Linear Programming is conditional
    - ✓ The solution will vary as the constraints (limiting conditions) change – then the corners of the feasible solution region also change

2. Linear Programming (LP)

- Graphical Interpretation of LP Problems
  - Conclusions:

1. The OF extremum is always found at the borders on the feasible region (FR) – and it is always at some of the corners.
2. The optimum is found at that corner in which the OF does not cross the feasible region
3. At each corner of the feasible region the so-called ***basic feasible solution*** is found



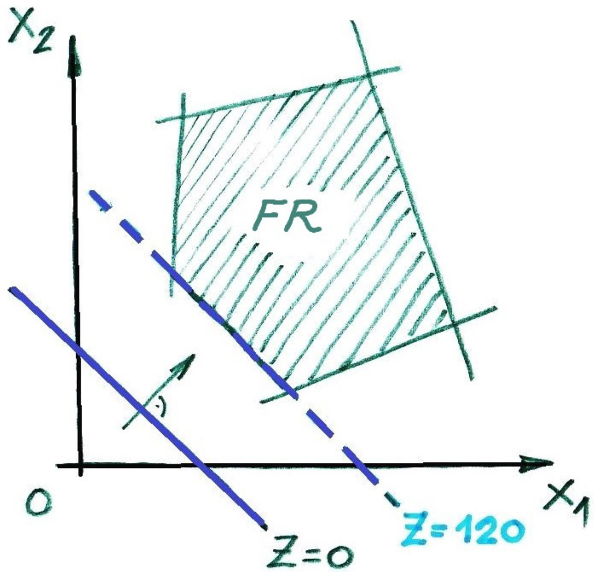
4. When we look for the optimum solution it is sufficient to find all corners of the feasible region (especially if FR it is ***bounded***)

Co-funded by the Erasmus+ Programme of the European Union

2. Linear Programming (LP)

- Graphical Interpretation of LP Problems
  - Conclusions:

5. The OF is at a corner of FR when the number of variables equal to zero is equal to the number of the decision variables k.
6. When the line of OF at $Z = 0$ (the intersection between the plane of OF and the plane $X_1 0 X_2$) is parallel to one of the borders of FR, this means that there is no single optimal solution.



7. It is possible to have feasible region, but not to find the optimum solution
   - In that case, either the constraint which is parallel to OF, or the OF have to be redefined.

2. Linear Programming (LP)

- There are 3 major types of LP problems
  - ➢ **Transportation Problem**
    - ✓ formulated in 1941 by L. Kantorovich (USSR) and F.L. Hitchcock (USA) independently one form another
    - ✓ During the WWII the transportation costs have to be minimized
  - ➢ **Diet Problem or Nutrition Problem**
    - ✓ also known as Mixing or Blending Problem
    - ✓ formulated in 1945 by G. Stigler (Great Britain)
  - ➢ **(Resource) Allocation Problem**
    - ✓ also known as Assignment Problem
    - ✓ formulated in 1947 G.B. Dantzig (USA)
  - ➢ Only Allocation and Transportation Problems are discussed
    - ✓ They have application in Water Resources Management

3. Allocation Problem

- The Allocation problem can be formulated using the following **example**
  - An agricultural farm has an area of F = 50 ha
  - Farmer is going to grow two crops - X_1 and X_2.
    - ✓ The profit from crop X_1 (e.g. Maize) is 300 €/ha
    - ✓ The profit from crop X_2 (e.g. Sunflower) is 200 €/ha
  - The fertilizers needed to obtain these profits are:
    - ✓ for crop X_1 (e.g. Maize) - 500 kg/ha
    - ✓ for crop X_2 (e.g. Sunflower) - 250 kg/ha
  - According to National requirements the average fertilizer rate per hectare cannot exceed 400 kg/ha, i.e. for an area of F = 50 ha the total amount of fertilizers for the farm is 50.400 = 20 000 kg/year.
  - Find areas X_1 and X_2 so to maximize farmer's profit.

3. Allocation Problem

- Mathematical formulation:
  - OF: $Z = 300X_1 + 200X_2 \rightarrow \max$
  - Constraints:
    - ✓ $X_1 + X_2 \leq 50$ (area constraint)
    - ✓ $500X_1 + 250X_2 \leq 20000$ (fertilizer constraint)
  - The second constrain maybe reduced to:
    - ✓ $2X_1 + X_2 \leq 80$ (fertilizer constraint)

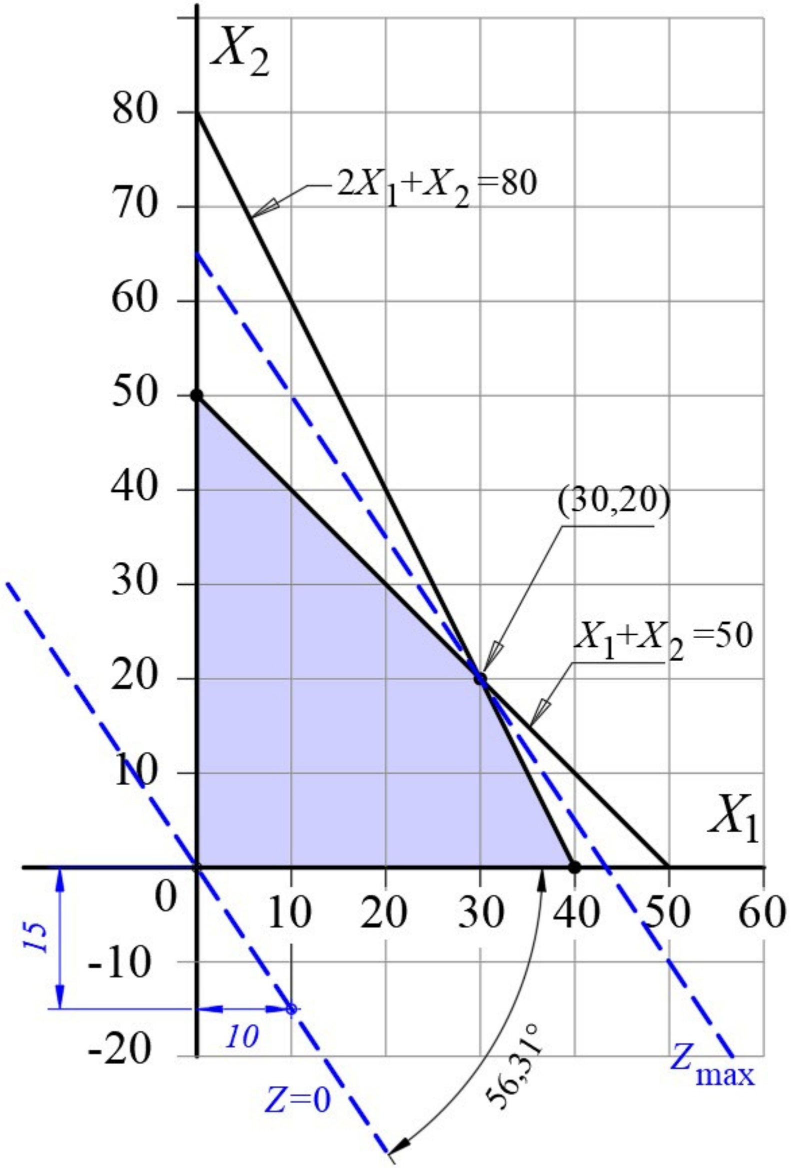
  - Note that the problem is in so-called ***canonical form***
    - ✓ The canonical form is found when the *maximum* of OF has to be obtained and all constraints are of type "*less or equal*"
    - ✓ Also, canonical form is present in case when the *minimum* of OF has to be obtained and all constraints are of type "*greater or equal*"

3. Allocation Problem

- Graphical Solution:
  - At first, the two constraints are presented as equations

$$X_1 + X_2 = 50 \text{ (area constraint)}$$
$$2X_1 + X_2 = 80 \text{ (fertilizer constraint)}$$

  - The equations represent lines in plane $X_1 0 X_2$
  - They can be drawn if two points of each line are found.
  - We can substitute consecutively X_1=0 and then X_2 =0, to find these points



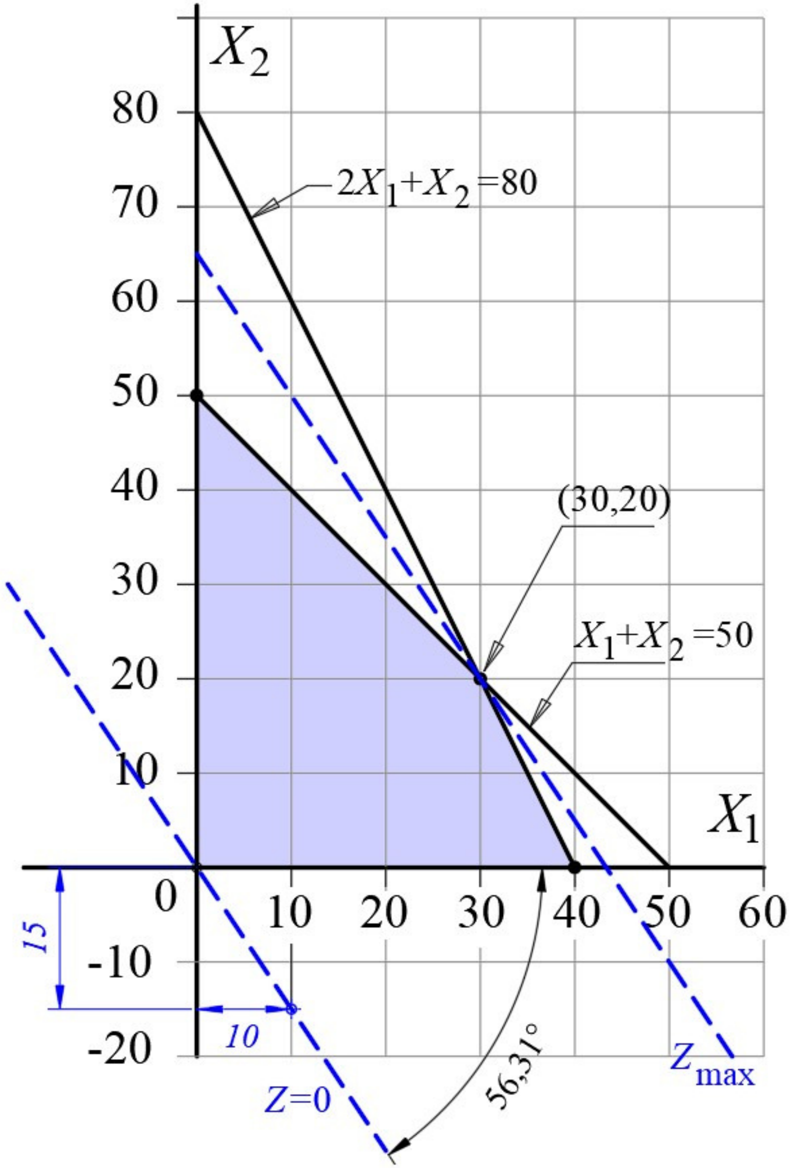
3. Allocation Problem

- Graphical Solution:
  - Second, because the constraints are actually inequalities, the half-planes containing feasible solutions have to be determined

    $X_1 + X_2 \leq 50$ (area constraint)

    $2X_1 + X_2 \leq 80$ (fertilizer constraint)

    - ✓ As it is seen from the inequalities of the two constraints, the origin of the coordinate system $X_1 0 X_2$ (point 0,0) is in the half-plane which satisfies these inequalities, thus the origin (0,0) shows the feasible half-plane.
  - Feasible region is found



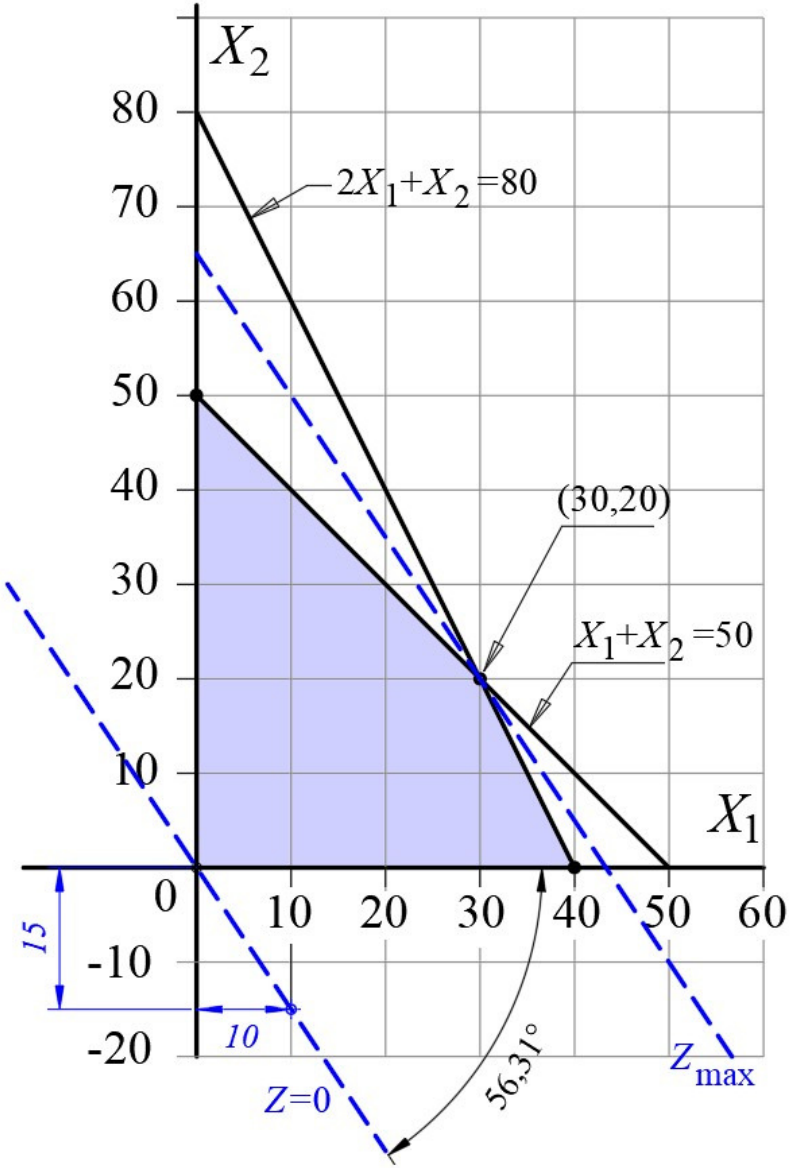
3. Allocation Problem

- Graphical Solution:
  - The line of the OF is drawn
  - The equation is

    $$Z = 300X_1 + 200X_2$$

  - If $Z = 0$, then the intersection line between the plane $X_1 0 X_2$ and the plane of the OF is found
  - Obviously, when $Z = 0$, the line of the OF intersects the origin 0,0.
  - The position of OF line in the plane $X_1 0 X_2$ is found by rearranging the equation $300X_1 + 200X_2 = 0$

    $$\frac{X_2}{X_1} = -\frac{300}{200}$$



Co-funded by the Erasmus+ Programme of the European Union

3. Allocation Problem

- Graphical Solution:
  - It is known that:

$$tg\alpha = \frac{X_2}{X_1}$$

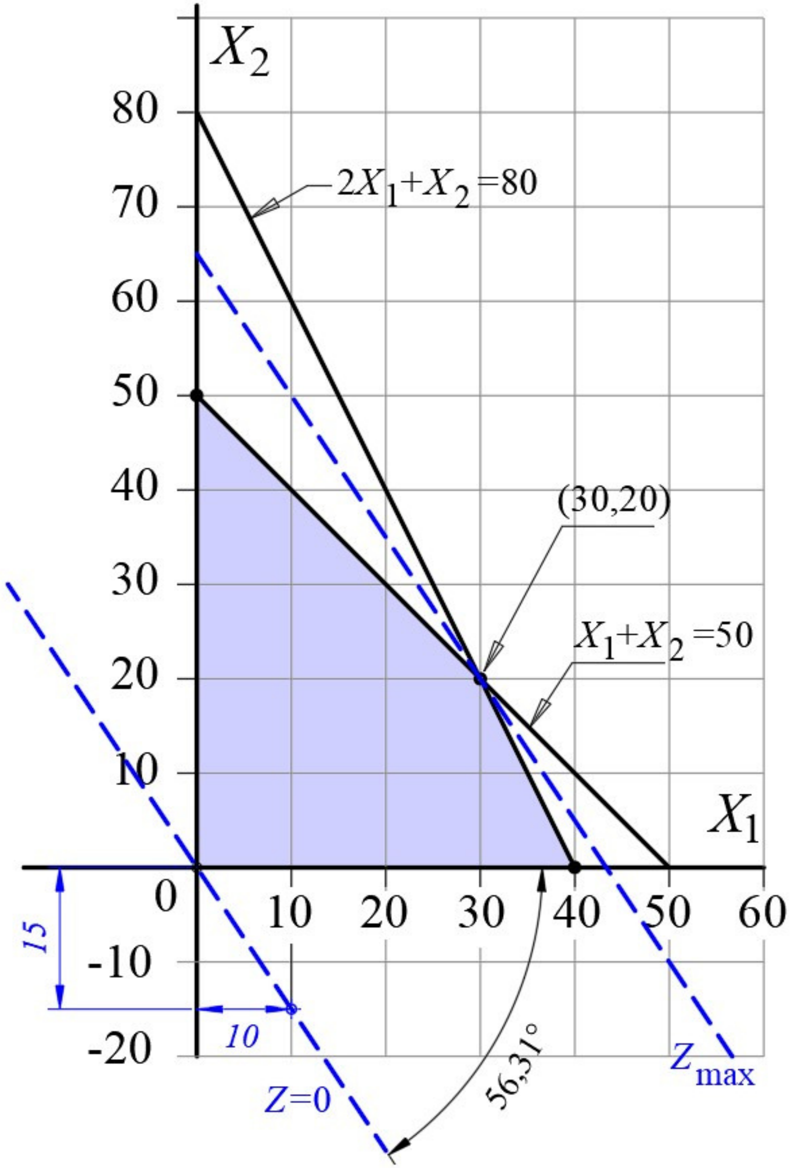
where α is the angle between the axis X_1 and the line of the OF

  - Thus we obtain:

$$tg\alpha = \frac{X_2}{X_1} = -\frac{300}{200} = -1,5$$

or $\alpha = -56,31°$

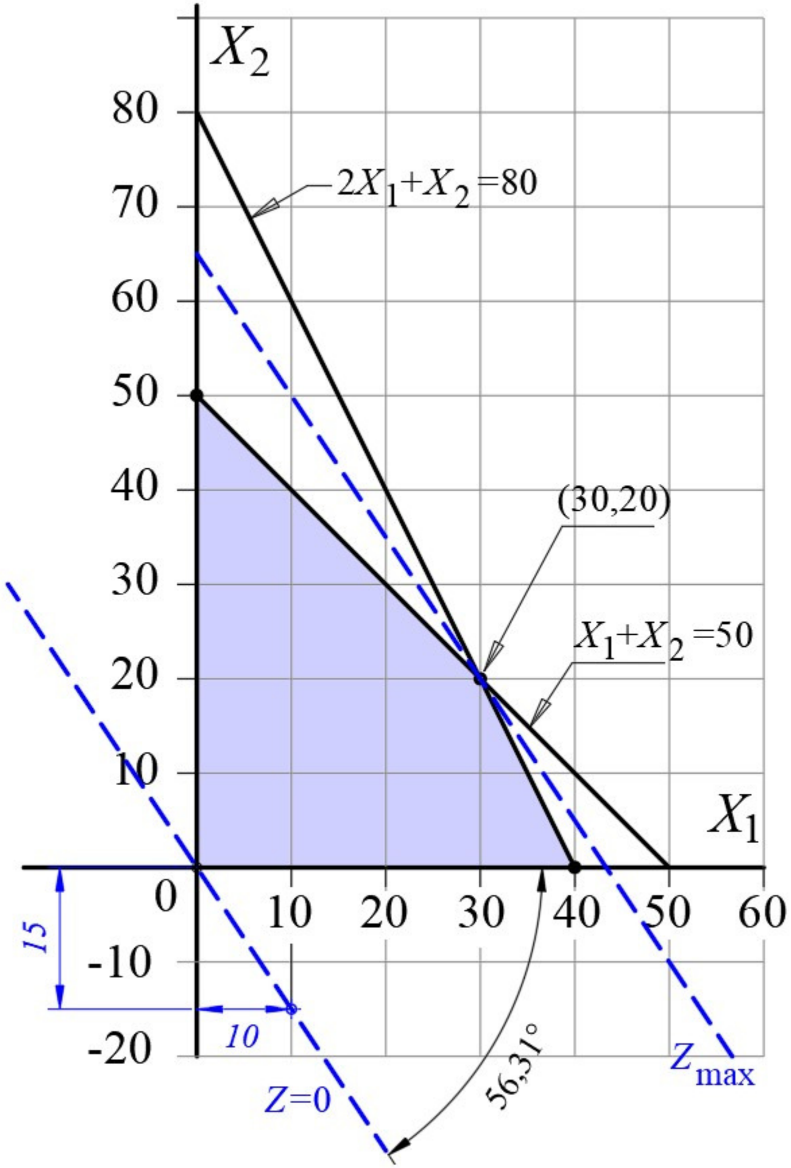
  - The position of the OF line when $Z = 0$ is found (see figure)



3. Allocation Problem

- Graphical Solution:
  - ➢ The optimal solution is found by translating the OF line in normal direction till it intersects that corner at which the line do not crosses the feasible region.
    - ✓ For the current example, this is the **point (30,20)**
  - ➢ Replacing $X_1 = 30$ and $X_2 = 10$ in the OF equation, we find:
  - ➢ $Z_{max} = 300.30 + 200.20 = 13000$ €.

N.B. One can try to estimate the OF value for any other point in the feasible region to see if the result is correct



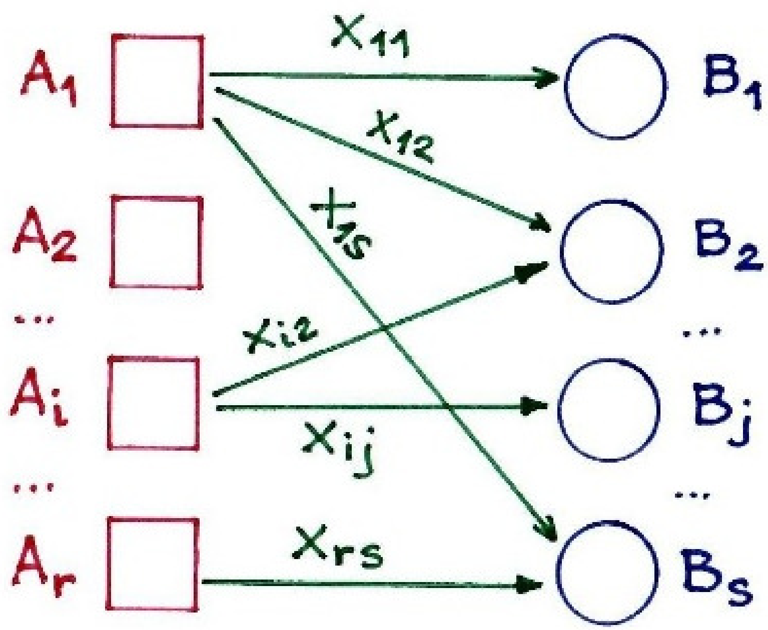
3. Allocation Problem

- Allocation problem in Water Resources Management
- Example
  - A reservoir is planned with a live storage V_{live} = 20 mil. m^3.
  - The reservoir can supply water for an Irrigation system and an Industrial complex, which will contribute to National Economy:
    - ✓ profit 0,1 €/m^3 from the Irrigation system;
    - ✓ profit 0,3 €/m^3 from the Industrial complex.
  - The energy consumption for delivery the water to users is:
    - ✓ to Irrigation System – 0,05 kWh/m^3.
    - ✓ to Industrial complex – 0,20 kWh/m^3.
  - The available energy for the Irrigation System and Industrial complex is estimated to 2,4 mil. kWh.
  - Find volumes of water to be supplied to two users.

4. Transportation Problem

- Definition of Transportation Problem
  - There are number of *r* producers A_i and number of *s* stores B_j.
  - Each producer A_i has it own capacity $W^A{}_i$ to supply given product to stores.
  - Each store A_i has its own capacity $W^B{}_j$ to receive the products.
  - The transported number of products from producer A_i to store B_j is X_{ij}.
  - The unit cost for transportation from producer A_i to store B_j is C_{ij}.

N.B. It is assumed that all products will be sold from stores B_j.



4. Transportation Problem

- Definition of Transportation Problem
  - The task is to find the *minimum cost* for transportation from producers A_i to stores B_j, i.e.
  - the **OF** is:

$$Z = X_{11}C_{11} + X_{12}C_{12} + X_{1s}C_{1s} + \ldots + X_{21}C_{21} + X_{22}C_{22} + \ldots + X_{r1}C_{r1} + \ldots + X_{rs}C_{rs},$$

or

$$Z = \sum_i \sum_j X_{ij} C_{ij}, \text{ where } i = 1 \div r; \quad j = 1 \div s.$$

  - Subject to constraints:

$$\sum_j X_{ij} \le W_i^A$$ - the capacity of each producer should not be exceeded

$$\sum_i X_{ij} \le W_j^B$$ - the capacity of each store should not be exceeded

N.B. It is typical for the problem to have constraints as equations, not as inequalities

4. Transportation Problem

- Types of Transportation Problem
  - **The problem is *balanced***
    - ✓ When the total production (supply) is equal to total demand:

$$\sum_i W_i^A = \sum_j W_j^B$$

  - **The problem is *unbalanced***
    - ✓ When the total production (supply) is not equal to total demand
    - ✓ It is possible to have a ***Surplus*** or

$$\sum_i W_i^A > \sum_j W_j^B$$

    - ✓ to have a ***Deficit***:

$$\sum_i W_i^A < \sum_j W_j^B$$

4. Transportation Problem

- Types of Transportation Problem
  - ➢ It is difficult, if not impossible to solve *unbalanced* problem
  - ➢ To make the problem *balanced*, it is introduced:
    - ✓ A ***dummy* store** (or *dump*), in case of a **surplus**
      - – The capacity of a dummy store is $W_{s+1}^{B} = \sum_{i} W_i^{A} - \sum_{j} W_j^{B}$
    - ✓ A dummy producer, in case of a deficit
      - – The capacity of a dummy producer is $W_{r+1}^{A} = \sum_{j} W_j^{B} - \sum_{i} W_i^{A}$
    - ✓ In general, the case of deficit is not typical for Transportation problem.
  - ➢ The unit costs for transportation *to* dummy store $C_{i,s+1}$ or *from* dummy producer $C_{r+1,j}$ is assumed ***enormously high***, compared to other unit costs.

4. Transportation Problem

- Objective Function
  - The OF $Z = \sum_{i} \sum_{j} X_{ij} C_{ij}$

is actually a product of two matrices:

Matrix of Quantities and Matrix of Unit Costs

	B_1	B_2	…	B_j	…	B_s
A_1	X_{11}	X_{12}		X_{1j}		X_{1s}
A_2	X_{21}	X_{22}		X_{2j}		X_{2s}
…						
A_i	X_{i1}	X_{i2}		X_{ij}		X_{is}
…						
A_r	X_{r1}	X_{r2}		X_{rj}		X_{rs}

	B_1	B_2	…	B_j	…	B_s
A_1	C_{11}	C_{12}		C_{1j}		C_{1s}
A_2	C_{21}	C_{22}		C_{2j}		C_{2s}
…						
A_i	C_{i1}	C_{i2}		C_{ij}		C_{is}
…						
A_r	C_{r1}	C_{r2}		C_{rj}		C_{rs}

✓ Usually the Unit Costs are constants, regardless of the quantities transported.

4. Transportation Problem

- The constraints
  - The constraints have to be converted to equations, if such exist
  - Constraints, regarding production (supply) will be

$$\left|\begin{array}{l} X_{11} + X_{12} + \ldots + X_{1s} = W_1^A \\ X_{21} + X_{22} + \ldots + X_{2s} = W_2^A \\ \ldots \\ X_{r1} + X_{r2} + \ldots + X_{rs} = W_r^A \end{array}\right.$$

  - Constraints, regarding stores (demand) will be:

$$\left|\begin{array}{l} X_{11} + X_{21} + \ldots + X_{r1} = W_1^B \\ X_{12} + X_{22} + \ldots + X_{r2} = W_2^B \\ \ldots \\ X_{1s} + X_{2s} + \ldots + X_{rs} = W_s^B \end{array}\right.$$

  - It is understandable, without saying that $X_{ij} \geq 0$.

4. Transportation Problem

- Variables
  - The number of constraints: $m = r + s - 1$
    - It is "*minus* 1", because one of the constraints is dependable, due to equality $\sum_i W_i^A = \sum_j W_j^B$
  - Variables: $n = r \times s$
    - If not all constraints are equations and the number of inequalities is t, then the number of *slack variables* will be t. In that case, the number of variables will be $n = r \times s + t$
  - Basic variables: m
  - Decision variables: $k = n - m = r \times s - (r + s - 1) = r\,(s-1) - (s-1)$

$$k = (r-1)(s-1)$$

  - Usually the number k is too big, or at least bigger than 2, to solve the Transportation problem graphically

4. Transportation Problem

- Solving the Transportation problem
- Example
  - There are 3 producers, capable to deliver the following production: $W_1^A = 50,\ W_2^A = 35,\ W_3^A = 15.$
  - There are 4 stores, with the following demands:
$$W_1^B = 25,\ W_2^B = 30,\ W_3^B = 20,\ W_4^B = 25$$
  - The minimum costs for transportation of production to the stores should be obtained, using the following unit costs:

	B_1	B_2	B_3	B_4
A_1	2	1	4	2
A_2	4	5	2	3
A_3	3	2	3	2

The cost to transport production from producer A_1 to store B_1 is 2 €/unit

  - All the constraints are equations

4. Transportation Problem

- <u>Example</u>
  - The problem is balanced, since:

$$\sum_i W_i^A = 100 \text{ and } \sum_j W_j^B = 100$$

  - Number of producers: $r = 3$;
  - Number of stores: $s = 4$.
  - Number of Variables: $n = r \times s = 3 \times 4 = 12$.
  - Number of Constraints: $m = r + s - 1 = 3 + 4 - 1 = 6$
  - Basic variables: $m = 6$
  - Decision variables: $k = (r-1)(s-1) = (3\text{-}1)(4-1) = 6$.

4. Transportation Problem

- Example
  - Finding the basic feasible solution
    - ✓ According to the conclusion from the graphical interpretation of LP, when a basic feasible solution is found, then the number of variables, which are zero is equal to the number of decision variables.
    - ✓ Thus, we have to assign value of zero to 6 variables – these are decision variables.
    - ✓ The values of the basic variables (in this example also 6) will be found by means of satisfying the constraints (which are $m = 6$).
  - Finding the basic feasible solution can be done by two methods:
    - ✓ Northwest corner method
    - ✓ Least Cost method

4. Transportation Problem

- Example
  - Finding the basic feasible solution by Northwest corner method
    - ✓The following table is prepared:

Matrix of Quantities

	B_1	B_2	B_3	B_4	W^A
A_1	25	25	0	0	50
A_2	0	5	20	10	35
A_3	0	0	0	15	15
W^B	25	30	20	25	100 = 100

  - Filling the table:
    - ✓ Starting from the top leftmost corner (Northwest corner) in the first cell the maximum possible quantity is assigned, which satisfies one of the constraints. In this example – 25 satisfies the demand of store B_1.
    - ✓ We move to cell A_1B_2, because the capacity of producer A_1 is not fulfilled. We assign 25 units to fulfill the capacity of producer A_1.

4. Transportation Problem

- Example
  - Finding the basic feasible solution by Northwest corner method

Matrix of Quantities

	B_1	B_2	B_3	B_4	W^A
A_1	25	25	0	0	50
A_2	0	5	20	10	35
A_3	0	0	0	15	15
W^B	25	30	20	25	100 = 100

- Filling the table:
  - ✓ Next step is to move down to cell A_2B_2 – the new Northwest corner. We assign 5 units in that cell to satisfy the demand of store B_2.
  - ✓ Then we move right to cell A_2B_3 and assign maximum possible quantity – 20, thus satisfying the demand of store B_3.
  - ✓ The procedure continues until all the constraints are fulfilled.

4. Transportation Problem

- Example
  - Finding the basic feasible solution by Northwest corner method

Matrix of Quantities

	B_1	B_2	B_3	B_4	W^A
A_1	25	25	0	0	50
A_2	0	5	20	10	35
A_3	0	0	0	15	15
W^B	25	30	20	25	100 = 100

	B_1	B_2	B_3	B_4
A_1	2	1	4	2
A_2	4	5	2	3
A_3	3	2	3	2

  - ✓ It is seen from the left table above that we have exactly 6 cells containing values of zero – these are the decision variables.
- OF value estimation:
  - ✓ Considering the Matrix of Quantities and the Matrix of Unit Costs:

$$Z = \Sigma X_{ij} C_{ij} = 25.2 + 25.1 + 5.5 + 20.2 + 10.3 + 15.2 = 160$$

4. Transportation Problem

- Example
  - Improvement of basic feasible solution
    - ✓ We select one basic variable and we make it a decision variable, and one of the decision variables becomes basic variable.
      - – This procedure is typical for the Simplex method for solving LP problems
    - ✓ In the Transportation problem, we select 4 cells which form rectangle or mini-matrix
    - ✓ A selection for the example is shown below

	B_1	B_2
A_1	25	25
A_2	0	5

- The cells may not be adjacent
- At least one value in the cells has to be zero
- It is possible to have zero values in two cells of that mini-matrix, but they should not be in one column or in one row

4. Transportation Problem

- <u>Example</u>
  - ➢ Improvement of basic feasible solution
    - ✓ For the selected mini-matrix we switch as big quantity as possible between columns or between rows (an example is shown below).

	B_1	B_2
A_1	⊖ 25	⊕ 25
A_2	⊕ 0	⊖ 5

– For the selected mini-matrix maximum possible transfer between columns is 5 units.

– This is how one variable becomes decision variable (it has value of 0) and a decision variable becomes basic (with value grater than zero)

✓We have to check if that switch increases or decreases the OF value

– if it increases the OF, then the switch is cancelled

– if it decreases the OF, it is accepted

Co-funded by the
Erasmus+ Programme
of the European Union

4. Transportation Problem

- Example
  - Improvement of basic feasible solution
    - ✓ The change in value can be estimated as follows:

$$\Delta Z = \Delta X.[(C_{12} + C_{21}) - (C_{11} + C_{22})],$$

where ΔX is the quantity switched (transferred)

C_{12} is the unit cost corresponding to the quantity delivered from the producer at the first row of this min-matrix to the store at the second column of the mini-matrix, etc.

- ✓ For the current example

$$\Delta Z = \Delta X.[(C_{12} + C_{21}) - (C_{11} + C_{22})] = 5.[(1 + 4) - (2 + 5)] = -10\ €$$

a) Matrix of Quantities

	B_1	B_2
A_1	20	30
A_2	5	0

b) Matrix of Unit Costs

	B_1	B_2
A_1	2	1
A_2	4	5

4. Transportation Problem

- <u>Example</u>
  - Finding the optimal solution
    - ✓ The switching of variables, i.e. the transfer of quantities from cell to cell in those mini-matrices is done in trial-and-error way
    - ✓ When the number of variables is large, this approach can take significant time
    - ✓ It is advisable to use some software
    - ✓ The Microsoft Excel has a build-in tool for solving optimization problems – it is Solver

N.B. If the task has dummy producer or dummy store, then the final value of the OF is found by subtracting the costs for transportation of quantities from dummy producer or to dummy store.

4. Transportation Problem

- <u>Transportation Problem in Water Resources Management</u>
  - ➢ The Transportation problem can also be found in WRM
  - ➢ Typical example is the task for determination of water delivery from multiple sources to multiple users.
    - ✓ If water sources are A_i and the water users are B_j, then the problem consists of finding the minimum cost for water delivery
  - ➢ This problem can be solved in two different cases:
    - ✓ Project case – during preliminary research for design of water supply network or irrigation network
      - – **Minimum investments in construction of the network is found**
    - ✓ Exploitation case – during exploitation stage of already existing networks, if there are canals/pipelines from each source to several (or all) water users
      - – **Minimum expenses for water delivery in a given period of time is found**

4. Transportation Problem

- Transportation Problem in Water Resources Management
  - ➢ The Transportation problem in WRM has its specifics
  - ➢ For the example of water delivery from multiple sources to multiple users.
    - ✓ In both cases – project and exploitation – the unit costs are NOT constant.
    - ✓ In project case – the canal/pipeline sizes depend on the design discharges – the bigger are the discharges, the higher are the costs
      - **But the relation *Q-Cost* is not linear!**
    - ✓ In exploitation case – the expenses for delivery depend on volumes – the bigger are the volumes, the higher are the expenses.
      - **In case of delivery by open canals – the expenses are almost constant or in approximately linear relation to volumes**
      - **In case of pressurized flow (by pumps) – the expenses are in non-linear relation to volumes**

4. Transportation Problem

- Transportation Problem in Water Resources Management
  - ➢ The Transportation problem in WRM can be solved using standard procedure, but taking into account the specifics
    - ✓ The OF is changed: $Z = \sum_i \sum_j Z_{ij}$

    where Z_{ij} are the total costs or total expenses, estimated as function of discharges or volumes

    - ✓ The functions $Z_{ij} - Q$ or $Z_{ij} - V$ have to be established
    - ✓ These functions can be polynomial, e.g.:

$$Z_{ij} = a_0 + a_1 V + a_2 V^2 + a_3 V^3 + a_4 V^4, €$$

More on solving Transportation problem in WRM is shown in task # 2.